A CHARACTERIZATION OF UNITARY DUALITY([1])

BY

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Abstract. The concept of unitary duality for topological groups was introduced by H. Chu. All mapping spaces are given the compact-open topology. Let G and H be locally compact groups. $G^\times$ is the space of continuous finite-dimensional unitary representations of G. Let $\mathrm{Hom}\,(G^\times, H^\times)$ denote the space of all continuous maps from $G^\times$ to $H^\times$ which preserve degree, direct sum, tensor product and equivalence. We prove that if H satisfies unitary duality, then $\mathrm{Hom}\,(G, H)$ and $\mathrm{Hom}\,(H^\times, G^\times)$ are naturally homeomorphic. Conversely, if $\mathrm{Hom}\,(Z, H)$ and $\mathrm{Hom}\,(H^\times, Z^\times)$ are homeomorphic by the natural map, where Z denotes the integers, then H satisfies unitary duality. In different contexts, results similar to the first half of this theorem have been obtained by Suzuki and by Ernest. The proof relies heavily on another result in this paper which gives an explicit characterization of the topology on $\mathrm{Hom}\,(G^\times, H^\times)$. In addition, we give another necessary condition for locally compact groups to satisfy unitary duality and use this condition to present an example of a maximally almost periodic discrete group which does not satisfy unitary duality.

Introduction. Let G be a locally compact topological group. Let $G^\times$ denote the space of continuous finite-dimensional unitary representations of G with the compact-open topology. $G^{\times\times}$ is the space of continuous unitary "representations" of $G^\times$ (see Definition 2 below), again with the compact-open topology. $G^{\times\times}$ becomes a topological group in a natural way and there is a canonical homomorphism from G to $G^{\times\times}$. If this canonical map is a topological isomorphism we say that G satisfies unitary duality. This definition was introduced by Chu [2]. It presents in a more general context the duality theorems of Pontryagin [9, p. 94 ff.], Tannaka [7], and Takahashi [6].

In §1 of this paper we give the important definitions related to unitary duality. $\mathrm{Hom}\,(G^\times, H^\times)$ for two locally compact groups G and H is defined in a natural way and given the compact-open topology also. In §2 we give an explicit subbase for the neighborhoods of each point in $\mathrm{Hom}\,(G^\times, H^\times)$. In §3 we use the result of §2 to prove the following characterization of unitary duality: if G and H are locally

compact groups and H satisfies unitary duality, then $\mathrm{Hom}\,(G, H)$ and $\mathrm{Hom}\,(H^\times, G^\times)$ are homeomorphic by the natural map. Conversely, if $\mathrm{Hom}\,(Z, H)$ and $\mathrm{Hom}\,(H^\times, Z^\times)$ are homeomorphic by the natural map, where Z denotes the integers, then H satisfies unitary duality. This is related to a theorem of Suzuki [5], who proved the first half of our theorem for compact groups, using the Tannaka duality theorem. The spaces and topologies in Suzuki's theorem appear to be slightly different from those used here, but they are equivalent. A similar theorem for the case of Tatsuuma duality [8] has been proved by Ernest in [4]. In §4 we present another necessary condition for a locally compact group to satisfy unitary duality, and we use this result to produce a maximally almost periodic locally compact (in fact, discrete) group which does not satisfy unitary duality.

1. **Definitions.** For further information about Definitions 1 and 2, the reader may consult [2].

DEFINITION 1. Let G be a locally compact group and $\mathcal{U}_n$ the n-dimensional (complex) unitary group. Then $G_n^\times = \mathrm{Hom}\,(G, \mathcal{U}_n)$ is the space of all continuous n-dimensional unitary representations of G with the compact-open topology. $G^\times$ is defined to be $\bigcup_n G_n^\times$ taken as the topological sum. $G^\times$ is a locally compact space closed under the operations $\oplus$ (direct sum), $\otimes$ (tensor product), and unitary equivalence of representations. For $D \in G_n^\times$, we write $d(D) = n$.

DEFINITION 2. A *unitary representation of* $G^\times$ is a continuous function $\tau : G^\times \to \mathcal{U} = \bigcup_n \mathcal{U}_n$ satisfying the following four conditions:

(1) $\tau(G_n^\times) \subset \mathcal{U}_n$,

(2) $\tau(D \oplus E) = \tau(D) \oplus \tau(E)$ for $D, E \in G^\times$,

(3) $\tau(D \otimes E) = \tau(D) \otimes \tau(E)$ for $D, E \in G^\times$,

(4) $\tau(PDP^{-1}) = P\tau(D)P^{-1}$ for $D \in G^\times$, $P \in \mathcal{U}_{d(D)}$.

The set $G^{\times\times}$ of all unitary representations of $G^\times$ is given the compact-open topology, and $G^{\times\times}$ forms a m.a.p. (maximally almost periodic) topological group under the multiplication given by $(\sigma\tau)(D) = \sigma(D)\tau(D)$. Apparently it is not known whether $G^{\times\times}$ is locally compact. We have a natural homomorphism $\mu_G : G \to G^{\times\times}$ given by $[\mu_G(x)](D) = D(x)$ for $D \in G^\times$ and $x \in G$. μ_G is automatically continuous and is injective if and only if G is m.a.p. If μ_G is a topological isomorphism, we say that *G satisfies unitary duality.*

DEFINITION 3. Let G and H be locally compact groups. We say that a continuous map $\sigma : G^\times \to H^\times$ is a *morphism* if the following four conditions are satisfied:

(1) $\sigma(G_n^\times) \subset H_n^\times$ for every positive integer n,

(2) $\sigma(D \oplus E) = \sigma(D) \oplus \sigma(E)$ for $D, E \in G^\times$,

(3) $\sigma(D \otimes E) = \sigma(D) \otimes \sigma(E)$ for $D, E \in G^\times$,

(4) $\sigma(PDP^{-1}) = P\sigma(D)P^{-1}$ for $D \in G^\times$, $P \in \mathcal{U}_{d(D)}$.

The set of these morphisms $\sigma : G^\times \to H^\times$ with the compact-open topology is denoted by $\mathrm{Hom}\,(G^\times, H^\times)$. It is obviously a Hausdorff space.

DEFINITION 4. Let $\mathscr{G}$ be the category of locally compact topological groups and $\mathscr{H}$ the category of Hausdorff topological groups, with continuous homomorphisms as morphisms in each case. Let $\mathscr{R}$ be the category in which the objects are $G^\times$ for G in $\mathscr{G}$, and the morphisms are defined by Definition 3. We define the contravariant functor $\varphi: \mathscr{G} \to \mathscr{R}$ by specifying $\varphi(G) = G^\times$ and for $f \in \operatorname{Hom}(G, H)$ we define $\varphi(f) = f^\times \in \operatorname{Hom}(H^\times, G^\times)$ by $f^\times(D) = D \circ f$. Similarly, we define the contravariant functor $\psi: \mathscr{R} \to \mathscr{H}$ by $\psi(G^\times) = G^{\times\times}$ and for $\sigma \in \operatorname{Hom}(G^\times, H^\times)$ we define $\psi(\sigma) = \sigma^\times \in \operatorname{Hom}(H^{\times\times}, G^{\times\times})$ by $\sigma^\times(\tau) = \tau \circ \sigma$. Using this notation, we have $\mu = \{\mu_G: G \to G^{\times\times}\}$ is a natural transformation from the embedding functor $\iota: \mathscr{G} \to \mathscr{H}$ to the functor $\psi \circ \varphi$.

We adopt the notation $N(K, V)$ for the set of all functions g in any specific set of functions such that $g(K) \subset V$.

2. Explicit characterization of the topology for $\operatorname{Hom}(G^\times, H^\times)$.

DEFINITION 5. Let $\sigma_0 \in \operatorname{Hom}(G^\times, H^\times)$, n a positive integer, K a subset of $G_n^\times$, F a subset of H, and V a subset of $\mathscr{U}_n$. Then $O(\sigma_0, K, F, V)$ is defined to be the subset of $\operatorname{Hom}(G^\times, H^\times)$ consisting of all σ such that $[(\sigma_0 D)x^{-1}][(\sigma D)x] \in V$ for every $D \in K$ and every $x \in F$.

THEOREM 1. *Let $\sigma_0 \in \operatorname{Hom}(G^\times, H^\times)$. As K runs through the compact subsets of $G_n^\times$, F runs through the compact subsets of H, V runs through a base of open neighborhoods of the identity in $\mathscr{U}_n$ and n runs through the positive integers, the family of all $O(\sigma_0, K, F, V)$ thus formed is a subbase for the open neighborhoods of σ_0 in the compact-open topology on $\operatorname{Hom}(G^\times, H^\times)$.*

Proof. We show first that each $O(\sigma_0, K, F, V)$ is open. Suppose $\sigma \in O(\sigma_0, K, F, V)$. For the moment let D denote a fixed element of $K \subset G_n^\times$. For each $x \in F$, let A_x be a neighborhood of $[\sigma_0(D)](x)$ in $\mathscr{U}_n$ and B_x a neighborhood of $[\sigma(D)](x)$ in $\mathscr{U}_n$ such that $A_x^{-1} B_x \subset V$. Then let L_x be a compact neighborhood of x in H such that $[\sigma_0(D)](L_x) \subset A_x$ and $[\sigma(D)](L_x) \subset B_x$. Then there is a finite set of L_x's which cover F, by the compactness of F. Let these be called $L_1, \ldots, L_t$ and let $A_1, \ldots, A_t$ and $B_1, \ldots, B_t$ be the A's and B's corresponding to the respective x's. By known properties of compact-open topologies for locally compact spaces, the evaluation map $(E, x) \to E(x)$ of $H_n^\times \times H \to \mathscr{U}_n$ is continuous. This means that for each i, $1 \leq i \leq t$, we can find compact neighborhoods P_i of $\sigma_0(D)$ and Q_i of $\sigma(D)$ such that $P_i(L_i)$ is a compact subset of A_i and $Q_i(L_i)$ is a compact subset of B_i, where $P_i(L_i)$ is the subset of $\mathscr{U}_n$ consisting of all $E(x)$ where $E \in P_i$ and $x \in L_i$, and $Q_i(L_i)$ is defined analogously. For the particular $D \in G_n^\times$ in question, we pick S_D to be a compact neighborhood of D in $G_n^\times$ so that $\sigma_0(S_D) \subset P_i$ and $\sigma(S_D) \subset Q_i$ for each i, $1 \leq i \leq t$. Now if we repeat the above procedure for each $D \in K$, we may extract a finite collection of S_D's which cover K, by the compactness of K. Let us call this collection $S_1, \ldots, S_r$, and then renumber the A's, B's, L's, P's, and Q's chosen above so that the ones with which S_j is associated are $A_{1j}, A_{2j}, \ldots, A_{t_j t}$, etc. Finally,

let

$$T = \bigcap_{j=1}^{r} \bigcap_{i=1}^{t_j} N(S_j, N(L_{ij}, B_{ij}));$$

T is an open subset of $\text{Hom}(G^\times, H^\times)$ and σ is in T. Now suppose $\tau \in T$, and let $D \in K$ and $x \in F$. Then $D \in S_j$ for some j, $1 \le j \le r$, and for this j, we have $x \in L_{ij}$ for some i, $1 \le i \le t_j$. Therefore

$$\sigma_0(D)(x)^{-1}\tau(D)(x) \in \sigma_0(S_j)(L_{ij})^{-1}\sigma(S_j)(L_{ij}) \subset P_{ij}(L_{ij})^{-1}Q_{ij}(L_{ij}) \subset A_{ij}^{-1}B_{ij} \subset V,$$

whence $\tau \in O(\sigma_0, K, F, V)$. This shows that $O(\sigma_0, K, F, V)$ is open.

On the other hand, let $\sigma_0 \in N(K, U) \subset \text{Hom}(G^\times, H^\times)$, where K is a compact subset of $G_n^\times$ and U is an open subset of $H_n^\times$, for some positive integer n. Now, $H_n^\times$ has the compact-open topology, so for a given $D \in K$ we may find a neighborhood of $\sigma_0(D)$ of the form $\bigcap_{i=1}^{t} N(F_i, W_i)$, where the F_i are compact subsets of H, and W_i are open subsets of $\mathcal{U}_n$, which is entirely contained inside U. Then we can let L be a compact neighborhood of D in $G_n^\times$ such that $\sigma_0(L) \subset \bigcap_{i=1}^{t} N(F_i, W_i)$. As D runs through the elements of K, L runs through a covering of K, so by compactness we may find a finite subcovering $L_1, \ldots, L_s$ of K, and then by reindexing we have compact subsets F_{ij} of H and open subsets W_{ij} of $\mathcal{U}_n$ $(1 \le j \le s, 1 \le i \le t_j)$ such that

$$\sigma_0(L_j) \subset \bigcap_{i=1}^{t_j} N(F_{ij}, W_{ij}) \subset U.$$

Then because $\sigma_0(L_j)$ is compact in $H_n^\times$, we see that $[\sigma_0(L_j)](F_{ij})$ is a compact subset of W_{ij} in H, for each i and j. Therefore we can select a neighborhood V_{ij} of the identity element in $\mathcal{U}_n$ such that $[\sigma_0(L_j)](F_{ij}) \cdot V_{ij} \subset W_{ij}$ for each i and j. Finally, let

$$T = \bigcap_{j=1}^{s} \bigcap_{i=1}^{t_j} O(\sigma_0, L_j, F_{ij}, V_{ij}).$$

Suppose $\sigma \in T$, and let $D \in K$. Then $D \in L_j$ for some j, $1 \le j \le s$. Suppose $x \in F_{ij}$ for this particular j and some i, $1 \le i \le t_j$. Then

$$\sigma_0(D)(x)^{-1}\sigma(D)(x) \in V_{ij};$$

hence

$$\sigma(D)(x) \in \sigma_0(D)(x) \cdot V_{ij} \subset [\sigma_0(L_j)](F_{ij}) \cdot V_{ij} \subset W_{ij}.$$

Therefore

$$\sigma(D) \in \bigcap_{i=1}^{t_j} N(F_{ij}, W_{ij}) \subset U.$$

Hence $\sigma \in N(K, U)$. This concludes the proof of the theorem.   Q.E.D.

3. A characterization of unitary duality.

THEOREM 2. *Let G and H be locally compact groups. If H satisfies unitary duality, then the map $\varphi_{G,H}$: $\text{Hom}(G, H) \to \text{Hom}(H^\times, G^\times)$ given by $\varphi_{G,H}(f) = f^\times$, where*

$f^\times(D) = D \circ f$ for $f \in$ Hom (G, H) and $D \in H^\times$, is a homeomorphism. Conversely, if $\varphi_{Z,H}:$ Hom $(Z, H) \to$ Hom $(H^\times, Z^\times)$ is a homeomorphism, where Z denotes the integers, then H satisfies unitary duality.

Proof. Assume first that H satisfies unitary duality. Let f_1 and f_2 be elements of Hom (G, H) with $f_1 \neq f_2$. Then $f_1(y) \neq f_2(y)$ for some $y \in G$. Since $H^{\times\times}$ is m.a.p. and $\mu_H: H \to H^{\times\times}$ is an isomorphism, H must be m.a.p. Therefore we can find $D \in H^\times$ such that $D(f_1(y)) \neq D(f_2(y))$; that is $f_1^\times(D) \neq f_2^\times(D)$. Therefore the map $f \mapsto f^\times$ is injective. Now let $\sigma \in$ Hom $(H^\times, G^\times)$. Let f be the element of Hom (G, H) given by $f = \mu_H^{-1} \circ \sigma^\times \circ \mu_G$. We verify that $f^\times = \sigma$. Let $D \in H^\times$ and $y \in G$. Then

$$[f^\times(D)](y) = (D \circ \mu_H^{-1} \circ \sigma^\times \mu_G)(y) = [\mu_G(y) \circ \sigma](D) = [\sigma(D)](y).$$

Therefore we have shown that the map $f \mapsto f^\times$ is bijective and the map $\sigma \mapsto \mu_H^{-1} \circ \sigma^\times \circ \mu_G$ is its inverse.

Now let C be the space of all continuous functions from G to H with the compact-open topology. Since H is a topological group, pointwise multiplication makes C a topological group also [1, p. 492]. Therefore neighborhoods of any point of C are just translates of neighborhoods of the identity element in C. Since Hom (G, H) is a subspace of C, we have just shown that for any fixed $f \in$ Hom (G, H), the set of all $M(f, F, V)$ for F a compact set in G and V an open neighborhood of the identity in H constitutes a base for the open neighborhoods of f in the compact-open topology on Hom (G, H), where by $M(f, F, V)$ we mean the set of all $g \in$ Hom (G, H) such that $f(x)^{-1}g(x) \in V$ for every $x \in F$. But by the unitary duality of H, we may describe its topology in terms of the compact-open topology on $H^{\times\times}$. In particular, a subbase for the neighborhood of the identity of H is given by all $\mu_H^{-1}(N(K, V))$, where K is a compact subset of $H_n^\times$, V is an open neighborhood of the identity of $\mathcal{U}_n$, and n is a positive integer. Therefore a subbase for neighborhoods of any point f in Hom (G, H) is given by all $M(f, F, \mu_H^{-1}(N(K, V)))$. Then

$$g \in M(f, F, \mu_H^{-1}(N(K, V))) \Leftrightarrow f(y)^{-1}g(y) \in \mu_H^{-1}(N(K, V)) \quad \text{for every } y \in F$$
$$\Leftrightarrow D(f(y))^{-1}D(g(y)) \in V \quad \text{for every } y \in f \text{ and } D \in K$$
$$\Leftrightarrow [f^\times(D)](y)^{-1}[g^\times(D)](y) \in V \quad \text{for every } y \in F \text{ and } D \in K$$
$$\Leftrightarrow g^\times \in O(f^\times, K, F, V).$$

So the map $\varphi_{G,H}$ is a bijection which carries subbase elements onto subbase elements and vice versa, and is a homeomorphism from Hom (G, H) onto Hom $(H^\times, G^\times)$.

For the converse, we note that $Z^\times$ is the topological sum of Hom $(Z, \mathcal{U}_n)$ for positive integers n, and each Hom $(Z, \mathcal{U}_n)$ is homeomorphic to $\mathcal{U}_n$ via the natural map $D \mapsto D(1)$. Since $\mathcal{U}$ is the topological sum of the $\mathcal{U}_n$, we have a homeomorphism $\xi: Z^\times \to \mathcal{U}$ obtained by piecing together all the individual homeomorphisms, and this map ξ is easily seen to preserve the operations of direct sum, tensor product, and equivalence. Using this we have a homeomorphism $\bar{\xi}:$ Hom $(G^\times, Z^\times) \to G^{\times\times}$

given by the formula $\xi(\sigma) = \xi \circ \sigma$. Now suppose that $\varphi_{Z,H}$ is a homeomorphism. Consider the homeomorphism ξ_H^{-1} from H to $\mathrm{Hom}\,(Z, H)$, given by specifying $[\xi_H^{-1}(y)](1) = y$, for $y \in H$. We now have a sequence of homeomorphisms

$$H \xrightarrow{\;\xi_H^{-1}\;} \mathrm{Hom}\,(Z, H) \xrightarrow{\;\varphi_{Z,H}\;} \mathrm{Hom}\,(H^\times, Z^\times) \xrightarrow{\;\xi\;} H^{\times\times}.$$

We claim that $\xi \circ \varphi_{Z,H} \circ \xi_H^{-1} = \mu_H$. For $y \in H$ and $D \in H^\times$, we have

$$\begin{aligned}
[(\xi \circ \varphi_{Z,H} \circ \xi_H^{-1})(y)](D) &= \xi([\varphi_{Z,H}(\xi_H^{-1}(y))](D)) \\
&= \xi(D \circ [\xi_H^{-1}(y)]) \\
&= D(y) = [\mu_H(y)](D).
\end{aligned}$$

Therefore μ_H is a topological isomorphism.   Q.E.D.

COROLLARY. *The functor* $\varphi \colon \mathscr{G} \to \mathscr{R}$ *is continuous in the sense that each* $\varphi_{G,H}$ *is a continuous function from* $\mathrm{Hom}\,(G, H)$ *to* $\mathrm{Hom}\,(H^\times, G^\times)$.

Proof. Let $N(K, V)$ be a neighborhood of the identity in $H^{\times\times}$, as in the proof of Theorem 2. The continuity of μ_H insures that $\mu_H^{-1}(N(K, V))$ is a neighborhood of the identity of H, whether or not H satisfies unitary duality, and the implication

$$g \in M(f, F, \mu_H^{-1}(N(K, V))) \Rightarrow g^\times \in O(f^\times, K, F, V)$$

still stands.   Q.E.D.

4. Another necessary condition and an example.

DEFINITION 6. Let G be a locally compact group. For any positive integer n, we define $^nG = \bigcap_{D \in G_n^\times} (\mathrm{Ker}\, D)$, the intersection of the kernels of all continuous n-dimensional unitary representations of G. ^nG is obviously a closed normal subgroup of G.

THEOREM 3. *Let* G *be a locally compact group which satisfies unitary duality. Let* V *be a neighborhood of the identity in* G. *Then for some positive integer* n, *we must have* $^nG \subset V$.

Proof. $\mu_G(V)$ is a neighborhood of the identity in $G^{\times\times}$. Therefore we must have

$$\bigcap_{i=1}^{k} N(K_i, W_i) \subset \mu_G(V)$$

for some choice of K_i and W_i, where each K_i is a compact subset of some $G_m^\times$ and W_i is an open neighborhood of the identity in $\mathscr{U}_m$ for that same m. Let n be an integer large enough so that each K_i is contained in some $G_m^\times$ with $m \leq n$. We claim now that $^nG \subset V$. Let $x \in {}^nG$ and let $D \in K_i$ for any i, $1 \leq i \leq k$. Then

$$[\mu_G(x)](D) = D(x) = 1_{d(D)} \in W_i;$$

hence $\mu_G(x) \in \mu_G(V)$. Since μ_G is injective, we must have $x \in V$.   Q.E.D.

We are now prepared to give an example of a discrete group which is m.a.p. but does not satisfy unitary duality. Let p be a prime integer. Let A be a cyclic

group of order p^2 with generator a, and B a cyclic group of order p with generator b. Then every automorphism of A is determined by specifying $a \mapsto a^n$ for some integer n which is not divisible by p. Thus the automorphism group Aut (A) has exactly $\varphi(p^2) = p(p-1)$ elements, and so it must have a subgroup of order p. Let the automorphism $a \mapsto a^r$ generate this subgroup. We now can construct a non-commutative metacyclic group G_p with generators a and b satisfying the relations

$$b^{-1}ab = a^r, \qquad b^p = a^{p^2} = e, \qquad p^2 = \text{order of } a.$$

(G_p is actually a semidirect product of A and B.) Hence G_p satisfies the hypotheses of Corollary (47.14) on p. 338 of [3], which says that every irreducible representation of G_p which is not one-dimensional is an induced representation from a one-dimensional representation of the subgroup A, and hence has dimension $p = [G_p:A]$. This forces $^{(p-1)}G_p$ to be equal to ^1G_p, the commutator subgroup of G_p, which is nontrivial by our construction.

Now let $G = \bigoplus_p G_p$, the "weak" direct sum of the G_p for all primes p (by this we mean the subgroup of the direct product of the G_p consisting of those elements which have as pth coordinate the identity element of G_p for all but a finite number of coordinates) provided with the discrete topology. Since each G_p is m.a.p., so is G. However, for each p we have

$$^{(p-1)}G \cap {}^{(p-1)}G_p = {}^{(p-1)}G_p \neq \{e\}$$

so by Theorem 3 using $\{e\}$ in place of V, G cannot satisfy unitary duality.

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